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RESEARCH ARTICLE

A remark on discontinuous games with asymmetric information and ambiguity

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Abstract We consider discontinuous games with asymmetric information and ambiguity (i.e., players have maximin preferences *à la* Gilboa and Schmeidler (1989)). It is shown that the existence of equilibria follows directly from the existence of Nash equilibria in every ex post game if all players are endowed with the maximin preferences. This is false for discontinuous games where players have Bayesian preferences as shown in He and Yannelis (2015a).

Keywords Discontinuous game · Asymmetric information · Ambiguity · Maximin expected utility

Mathematics Subject Classification C62 · D81 · D82

Contents

[1] For some recent references on discontinuous games, see the symposium Reny (2016a), and also Carmona (2016), Carmona and Podczeck (2016), Flesch and Predtetchinski (2016), He and Yannelis (2016b), Nessan and Tian (2016), Prokopovych (2016), Reny (2016b, c) and Scalzo (2016).

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1 Introduction

Games with incomplete information have been extensively studied and found wide applications in many economics fields.[1] Bayesian games with discontinuous payoffs arise naturally in various applied works, including location games, auctions and price competitions. The Bayesian paradigm has been constantly criticized since Ellsberg (1961), and the non-expected utility theory has received much attention. The purpose of this note was to study the equilibrium existence problem in discontinuous games under incomplete information and ambiguity.

In the framework of Bayesian preferences, some results have been presented to guarantee the existence of pure/behavioral strategy equilibria in discontinuous games with incomplete information (see He and Yannelis 2015a, 2016a; Carbonell-Nicolau and McLean 2015), which typically build on the equilibrium existence result of Reny (1999) for discontinuous games with complete information. However, as demonstrated by counterexamples in He and Yannelis (2015a), the equilibrium existence result in the incomplete information framework is not a straightforward adaptation of the result of Reny (1999). In order to generalize the result of Reny (1999) to asymmetric information, one has to introduce some exogenous assumptions.

In this note, we adopt the maximin expected utility of Gilboa and Schmeidler (1989) (see also de Castro and Yannelis 2009).[2] The main result shows that by working with the maximin preferences, the existence of equilibria in games with incomplete information follows directly from the existence of equilibria for every ex post game.[3] As a result, the maximin framework solves the equilibrium existence issue without introducing any additional conditions. To demonstrate the usefulness of the main result, we present a timing game with asymmetric information as an illustrative example, which has an equilibrium when players have maximin preferences, but has no equilibrium when the Bayesian reasoning is adopted.

The rest of this note is organized as follows: Section 2 introduces the model of discontinuous games with asymmetric information. Section 3 reviews the result on deterministic discontinuous games and presents the main result of this note on the existence of equilibria when players adopt maximin preferences. An illustrative example is provided in Sect. 4.

2 Discontinuous games with asymmetric information

We consider an asymmetric information game

$$G = \{\Omega, (u_i, X_i, \mathcal{F}_i)_{i \in I}\}.$$

- The set of players is $I = \{1, 2, \ldots, N\}$.

[2] For some recent applications of maximin preference in general equilibrium theory and game theory, see, for example, de Castro et al. (2011), Angelopoulos and Koutsougeras (2015), de Castro et al. (2015), He and Yannelis (2015b), Liu (2015) and Guo and Yannelis (2016).

[3] An ex post game means the realized normal form game at some state. For the precise definition, see Sect. 2.

- The space Ω contains countably many states, which represent the uncertainty of the world. Let $\mathcal{F}$ be the power set of Ω.
- For each $i \in I$, $\mathcal{F}_i$ is a partition of Ω, which denotes the private information of player i. Let $\mathcal{F}_i(\omega)$ be the element of $\mathcal{F}_i$ including the state ω.
- Player i's action space X_i is a nonempty, compact and convex subset of a topological vector space. Denote $X = \prod_{i\in I} X_i$.
- The mapping $u_i : X \times \Omega \to \mathbb{R}$ is a random utility function representing the (ex post) preference of player i.

A random strategy of player i is a function x_i from Ω to X_i. The set of player i's random strategies is denoted by $\mathcal{L}_i = X_i^{\Omega}$. Let $\mathcal{L} = \prod_{i\in I} \mathcal{L}_i$. A game G is called compact if u_i is bounded for ever $i \in I$, that is, $\exists M > 0$, $|u_i(x, \omega)| \leq M$ for every $x \in X$, $\omega \in \Omega$ and $i \in I$. A game G is said to be quasiconcave (resp. concave) if $u_i(\cdot, x_{-i}, \omega)$ is quasiconcave (resp. concave) for every $x_{-i} \in X_{-i}$, $\omega \in \Omega$ and $i \in I$. For every $\omega \in \Omega$, $G_\omega = (u_i(\cdot, \omega), X_i)_{i\in I}$ is called an ex post game.

3 Existence of equilibrium under ambiguity

3.1 Deterministic case

We shall first review the results on discontinuous games with complete information.

Let $G_d = (X_i, u_i)_{i=1}^N$ denote a deterministic discontinuous game, that is, Ω is a singleton set. Given $x \in X$, let $u(x) = (u_1(x), \ldots, u_N(x))$ be the payoff vector of the game G_d. Define $\Gamma_d = \{(x, u(x)) \in X \times \mathbb{R} : x \in X\}$, which is the graph of the payoff vector $u(\cdot)$. Then $\overline{\Gamma_d}$ denotes the closure of Γ_d.

Definition 1 1. In the game G_d, player i can secure a payoff $\alpha \in \mathbb{R}$ at $(x_i, x_{-i}) \in X_i \times X_{-i}$ if there is a point $\overline{x_i} \in X_i$ such that $u_i(\overline{x_i}, y_{-i}) \geq \alpha$ for all y_{-i} in some open neighborhood of x_{-i}.
2. The game G_d is payoff secure if for every $i \in I$, every $(x_i, x_{-i}) \in X_i \times X_{-i}$, and any $\epsilon > 0$, player i can secure a payoff

$$(u_i(x_i, x_{-i}) - \epsilon, \ldots, u_i(x_i, x_{-i}) - \epsilon)$$

at $(x_i, x_{-i}) \in X_i \times X_{-i}$.
3. The game G_d is better-reply secure if whenever $(x^*, \alpha^*) \in \overline{\Gamma_d}$ and x^* is not a Nash equilibrium, then some player j can secure a payoff strictly above α_i^* at x^*.

Definition 2 A game G_d is reciprocal upper semicontinuous if for any $(x, \alpha) \in \overline{\Gamma_d} \setminus \Gamma_d$, there is a player i such that $u_i(x) > \alpha_i$.

Reny (1999) shows that a Nash equilibrium exists in a deterministic discontinuous game under the better-reply security property together with some regularity conditions. In addition, a game with the conditions of payoff security and reciprocal upper semicontinuity is better-reply secure.

Fact 1 [Reny (1999)] *Every compact, quasiconcave and better-reply secure deterministic game has a Nash equilibrium.*

3.2 General case under ambiguity

In the following, we shall consider discontinuous games with asymmetric information. Suppose that players could have multiple priors and are ambiguous. We follow the non-expected utility approach by adopting the notion of maximin preferences of Gilboa and Schmeidler (1989). For each player i, let M_i be the set of his possible priors such that for any $\pi_i, \pi_i' \in M_i$, $\pi_i(E) = \pi_i'(E)$ for any $E \in \mathcal{F}_i$. That is, priors must be consistent with each other on player i's private information partition. Without loss of generality, we assume that $\pi_i(E) > 0$ for any $E \in \mathcal{F}_i$ and $\pi_i \in M_i$. Given a strategy profile $f \in \mathcal{L}$, the maximin expected utility (MEU) of player i is

$$V_i(f) = \inf_{\pi_i \in M_i} \sum_{\omega \in \Omega} u_i(f(\omega), \omega)\pi_i(\omega).$$

The ex ante game is denoted by $G_0 = (V_i, \mathcal{L}_i)_{i \in I}$.

Definition 3 1. When players have maximin preferences, a strategy profile $f \in \mathcal{L}$ is said to be an equilibrium if it is a Nash equilibrium in the game G_0.
2. Suppose that M_i is a singleton set, and player i is restricted to choose f_i which is measurable with respect to $\mathcal{F}_i$ for each $i \in I$. Then f is said to be a Bayesian equilibrium if it is a Nash equilibrium in the ex ante game.

Remark 1 If M_i is a singleton set for each agent i, then the maximin expected utility above reduces to the standard Bayesian expected utility. If M_i is the set of all probability measures on $\mathcal{F}$ which agree with each other on $\mathcal{F}_i$, then it is the maximin expected utility considered in de Castro and Yannelis (2009).

In games with maximin preferences, priors must be consistent on the information partition of each player. The information asymmetry is captured by the MEU, and hence it is natural to relax the restriction of private information measurability. On the contrary, the information asymmetry in a Bayesian model is captured by the assumption of private information measurability of the strategy set of each player, i.e., each f_i is assumed to be private information measurable. If the private information measurability condition is relaxed in the Bayesian setup, then the game is reduced to be symmetric information.

It is demonstrated via counterexamples in He and Yannelis (2015a) that a Bayesian equilibrium may not exist in a discontinuous game with Bayesian preferences. They resolved this issue by proposing the "finite payoff security" condition. The following result shows that if we adopt the maximin preferences, then the existence of equilibria in the ex ante game follows immediately from the conditions that could guarantee the existence of Nash equilibria in ex post games.

Proposition 1 *If an asymmetric information game G is compact and quasiconcave, every ex post game G_ω is better payoff secure, and players are maximin preference maximizes, then there exists an equilibrium in the ex ante game G_0.*

Proof Since the ex post game G_ω is compact, quasiconcave and better-reply secure, there exists a Nash equilibrium $f(\omega)$ in G_ω by Fact 1. We claim that f is an equilibrium in the ex ante game G_0.

Suppose otherwise. Then there exists some player i and strategy g_i such that $V_i(f) < V_i(g_i, f_{-i})$. There exists a prior $\pi_i \in M_i$ such that

$$\begin{aligned}\sum_{\omega\in\Omega} u_i(f(\omega), \omega)\pi_i(\omega) &< \inf_{\pi_i'\in M_i} \sum_{\omega\in\Omega} u_i(g_i(\omega), f_{-i}(\omega), \omega)\pi_i'(\omega)\\ &\leq \sum_{\omega\in\Omega} u_i(g_i(\omega), f_{-i}(\omega), \omega)\pi_i(\omega),\end{aligned}$$

which implies that there exists a state $\omega_1 \in E$ such that $\pi_i(\omega_1) > 0$ and

$$u_i(f(\omega_1), \omega_1) < u_i(g_i(\omega_1), f_{-i}(\omega_1), \omega_1).$$

This is a contradiction. Therefore, f is an equilibrium of G_0. □

Remark 2 We would like to emphasize that in the setting where players adopt the Bayesian preferences and each ex post game is compact, quasiconcave and better payoff secure, even if we do not require the private information measurability for any player's strategy, Reny (1999)'s theorem is still not applicable to conclude the existence of an equilibrium in the ex ante game. Indeed, He and Yannelis (2015a) show that the condition that every ex post game is quasiconcave is not sufficient to guarantee the quasiconcavity of the ex ante game.

4 Timing games with asymmetric information

We study a class of two-person, non-zero-sum, noisy timing games with asymmetric information. Such games can be used to model behavior in duels as well as in R&D and patent races. Let G be an asymmetric information timing game. The state space is Ω. For player i, the information partition is denoted as $\mathcal{F}_i$ and the private prior π_i is defined on $\mathcal{F}_i$. The action space for both players is [0, 1]. At state ω, the payoff of player i is given by

$$u_i(a_i, a_{-i}, \omega) = \begin{cases} p_i(x_i, \omega), & \text{if } x_i < x_{-i} \\ q_i(x_i, \omega), & \text{if } x_i = x_{-i} \\ h_i(x_{-i}, \omega), & \text{otherwise} \end{cases}.$$

Suppose that the following conditions hold for $i = 1, 2$, $\omega \in \Omega$ and $x \in [0, 1]$:

1. $p_i(\cdot, \omega)$ and $h_i(\cdot, \omega)$ are both continuous and $p_i(\cdot, \omega)$ is nondecreasing,
2. $q_i(x, \omega) \in \text{co}\{p_i(x, \omega), h_i(x, \omega)\}$,[4]

[4] The notation co(A) denotes the convex hull of the set A.

3. if $q_i(x,\omega)+q_{-i}(x,\omega) < p_i(x,\omega)+h_{-i}(x,\omega)$, then $\mathrm{sgn}(p_i(x,\omega)-q_i(x,\omega)) = \mathrm{sgn}(q_{-i}(x,\omega)-h_{-i}(x,\omega))$.[5]

As shown in Reny (1999), each ex post game is compact, quasiconcave and payoff secure. We claim that each ex post game is reciprocal upper semicontinuous. If $q_i(x,\omega)+q_{-i}(x,\omega) \le p_i(x,\omega)+h_{-i}(x,\omega)$, then we have that $\mathrm{sgn}(p_i(x,\omega)-q_i(x,\omega)) = \mathrm{sgn}(q_{-i}(x,\omega)-h_{-i}(x,\omega))$. This case has already been shown in Reny (1999); we only need to consider the case that $q_i(x,\omega)+q_{-i}(x,\omega) > p_i(x,\omega)+h_{-i}(x,\omega)$. The reciprocal upper semicontinuity in the latter case is obvious since there must be some $i \in \{1,2\}$ such that $q_i(x,\omega) > p_i(x,\omega)$ or $q_i(x,\omega) > h_i(x,\omega)$. Therefore, if the conditions above hold and players are maximin preference maximizers, then this asymmetric information timing game has an ex ante equilibrium due to Proposition 1. The following example shows that an asymmetric information timing game may not possess an equilibrium if players have Bayesian preferences. However, this example has an equilibrium when all players have maximin preferences.

Example 1 [Nonexistence of Bayesian equilibria] The state space is $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, where

$$\omega_1 = \left(\frac{1}{2}, \frac{1}{2}\right),\ \omega_2 = \left(\frac{1}{2}, 1\right),\ \omega_3 = (1,1),\ \omega_4 = \left(1, \frac{1}{2}\right).$$

The information partitions are

$$\mathcal{F}_1 = \{\{\omega_1,\omega_2\},\{\omega_3,\omega_4\}\},\ \mathcal{F}_2 = \{\{\omega_1,\omega_4\},\{\omega_2,\omega_3\}\}.$$

The ex post utility functions of players at state $\omega = (t_1, t_2)$ are given as in the general model, where $p_i(x,\omega) = x - t_i$, $h_i(x,\omega) \equiv 0$ and

$$q_i(x,\omega) = \begin{cases} x - t_i, & \text{if } t_i < t_{-i}; \\ \frac{x-t_i}{2}, & \text{if } t_i = t_{-i}; \\ 0, & \text{if } t_i > t_{-i}. \end{cases}$$

Players 1 and 2 hold the common prior

$$\pi(\omega_1) = \pi(\omega_2) = \pi(\omega_3) = \frac{1}{3}, \quad \pi(\omega_4) = 0.$$

It is easy to see that this game satisfies all the specified conditions, and hence by Proposition 1, it possesses an equilibrium when both players are maximin preference maximizers. We claim that there is no Bayesian equilibrium in this game. By way of contradiction, suppose that (x_1, x_2) is a Bayesian equilibrium.

We shall first show that $x_i(\omega) \ge t_i$ at state $\omega = (t_1, t_2)$ for $i = 1, 2$. It is clear that $x_1(\omega), x_2(\omega) \ge \frac{1}{2}$ for any $\omega \in \Omega$; hence we only need to show $x_1(\omega_3) =$

[5] Notice that this condition is slightly weaker than the corresponding condition in Example 3.1 of Reny (1999). Example 1 satisfies our condition, but does not satisfy the condition of Reny (1999).

$x_1(\omega_4) = 1$ and $x_2(\omega_2) = x_2(\omega_3) = 1$. Suppose that $x_1(\omega_3) = x_1(\omega_4) < 1$. If $x_2(\omega_3) < x_1(\omega_3)$, then player 2 gets a negative payoff at the event $\{\omega_2, \omega_3\}$, and he can choose $x_2(\omega_2) = x_2(\omega_3) = 1$ to be strictly better off. If $x_2(\omega_3) \geq x_1(\omega_3)$, then player 1 gets a negative payoff at the event $\{\omega_3, \omega_4\}$, and he can choose $x_1(\omega_3) = x_1(\omega_4) = 1$ to be strictly better off. Thus, $x_1(\omega_3) = x_1(\omega_4) = 1$. Similarly, we can check that $x_2(\omega_2) = x_2(\omega_3) = 1$, as player 2 will otherwise get a negative payoff at the event $\{\omega_2, \omega_3\}$.

Now we consider the choice of player 2 at state ω_1.

1. If $x_2(\omega_1) = \frac{1}{2}$, then the best response of player 1 at the event $\{\omega_1, \omega_2\}$ is to choose the strategy $x_1(\omega_1) = x_1(\omega_2) = 1$. However, in this case, there is no best response for player 2 at the state ω_1.
2. If $x_2(\omega_1) = 1$, then there is no best response for player 1 at the event $\{\omega_1, \omega_2\}$.
3. Suppose that $x_2(\omega_1) = a \in (\frac{1}{2}, 1)$. If $x_1(\omega_1) = x_1(\omega_2) \in [\frac{1}{2}, a)$, then player 1 can always slightly increase his strategy to be strictly better off. If $x_1(\omega_1) = x_1(\omega_2) = a$, then player 1 can always slightly decrease his strategy to be strictly better off. If $x_1(\omega_1) = x_1(\omega_2) \in (a, 1]$, then the best response of player 1 must be $x_1(\omega_1) = x_1(\omega_2) = 1$, which implies that there is no best response for player 2 as shown in point (1).

Therefore, there is no Bayesian equilibrium.

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